

Queen Mary, University of London

B.Sc. EXAMINATION

MAS204 Calculus III

17th May, 2001: 10.00-13.00

This paper has two sections and you should attempt both sections. Please read carefully the instructions given at the beginning of each section.

Calculators may be used in this examination, but any programming, graph plotting or algebraic facility may not be used. Please state on your answer book the name and type of machine used.

You are reminded of the following, which you may use without proof.
In orthogonal curvilinear coordinates (u_1, u_2, u_3) , with corresponding unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and arc-length parameters h_1, h_2, h_3 , the gradient of a scalar field f is given by

$$\nabla f = \frac{1}{h_1} \frac{\partial f}{\partial u_1} \mathbf{e}_1 + \frac{1}{h_2} \frac{\partial f}{\partial u_2} \mathbf{e}_2 + \frac{1}{h_3} \frac{\partial f}{\partial u_3} \mathbf{e}_3.$$

The divergence of a vector field $\mathbf{F} = F_1 \mathbf{e}_1 + F_2 \mathbf{e}_2 + F_3 \mathbf{e}_3$ is given by

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} (h_2 h_3 F_1) + \frac{\partial}{\partial u_2} (h_3 h_1 F_2) + \frac{\partial}{\partial u_3} (h_1 h_2 F_3) \right],$$

and the curl of the same vector field is given by

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \mathbf{e}_1 & h_2 \mathbf{e}_2 & h_3 \mathbf{e}_3 \\ \partial/\partial u_1 & \partial/\partial u_2 & \partial/\partial u_3 \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}.$$

In spherical polar coordinates $(u_1, u_2, u_3) \equiv (r, \theta, \phi)$, the arc-length parameters are $h_1 = 1, h_2 = r, h_3 = r \sin \theta$.

In cylindrical polar coordinates $(u_1, u_2, u_3) \equiv (r, \theta, z)$, the arc-length parameters are $h_1 = 1, h_2 = r, h_3 = 1$.

SECTION A

Each question carries 8 marks. You should attempt ALL questions.

- A1.** (a) Evaluate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the vector field $\mathbf{F} = 3z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$ along the curve C on which $(x, y, z) = (t, t^2, t^3)$ from $t = 0$ to $t = 1$.
(b) Calculate ∇f for $f = y^2z + x^2y$ and evaluate it at $(2, 1, 3)$.

- A2.** Calculate $\nabla \cdot \mathbf{F}$ and $\nabla \times \mathbf{F}$ for

$$\mathbf{F} = \frac{1}{r}\mathbf{e}_\theta + r \cos \theta \mathbf{e}_\phi.$$

in spherical polar coordinates.

- A3.** Using index notation, or otherwise,

- (a) Prove that $\nabla(\mathbf{a} \cdot \mathbf{r}) = \mathbf{a}$ where $\mathbf{a}$ is a constant vector and $\mathbf{r}$ is the usual position vector (with components (x, y, z) in Cartesian coordinates).
(b) Prove that $\nabla \cdot (\nabla \times \mathbf{F}) = 0$ for any vector field $\mathbf{F}$.

- A4.** State the Divergence Theorem.

Using this theorem, or otherwise, evaluate $\int_S \mathbf{F} \cdot d\mathbf{S}$ for $\mathbf{F} = (x+y)\mathbf{i} + (y+z)\mathbf{j} + (z+x)\mathbf{k}$ over the surface S of the hemisphere defined by $x^2 + y^2 + z^2 \leq 9$, $z \geq 0$.

- A5.** For each of the following vector fields either find the general solution V of $\mathbf{F} = \nabla V$ or give a reason why no such V exists.

- (a) $\mathbf{F} = 2x\mathbf{i} + y^2\mathbf{j} + xz\mathbf{k}$.
(b) $\mathbf{F} = (1/r + \cos \theta)\mathbf{e}_r + \sin \theta \mathbf{e}_\theta$ in cylindrical polar coordinates.

- A6.** Show that at a general point the following are solutions of Laplace's equation $\nabla^2 V = 0$.

- (a) $V = r^n \cos n\theta$, for an integer n , in cylindrical polar coordinates.
(b) $V = r \sin \theta \cos \phi$, in spherical polar coordinates.

- A7.** For each of the following velocity fields, draw some representative streamlines:

- (a) $\mathbf{u} = (\sin y)\mathbf{i}$,
(b) $\mathbf{u} = (-y\mathbf{i} + x\mathbf{j})/r^2$, where $r^2 = x^2 + y^2$.

- (c) Determine whether the second flow is irrotational and whether it is incompressible.

SECTION B

Each question carries 22 marks. You may attempt all questions but only marks for the best 2 questions will be counted.

- B1.** Write down the form of Laplace's equation $\nabla^2 V = 0$ in cylindrical polar coordinates. Derive all the separable solutions of this equation of the form $V = R(r)S(\theta)$. Hence show that

$$V(r, \theta) = (A_0 \ln r + B_0)(C_0 \theta + D_0) + \sum_{m=1}^{\infty} \left(A_m r^m + B_m r^{-m} \right) \left(C_m \sin m\theta + D_m \cos m\theta \right).$$

is a solution of this equation independent of z , where A_m , B_m , C_m and D_m are constants for each m .

Consider the region D defined by $a \leq r \leq b$, $0 \leq \theta \leq \pi$, $-\infty < z < \infty$. Sketch the region in a plane perpendicular to the z -axis which lies in D . On the boundaries $r = a$, $\theta = 0$ and $\theta = \pi$, $V = 0$ while on the boundary $r = b$, $V = \theta \sin \theta$. Find the solution V of Laplace's equation in D , independent of z , which satisfies these boundary conditions.

[You may assume that on $0 \leq \theta \leq \pi$

$$\theta \sin \theta = \sum_{k=1}^{\infty} \frac{16k}{\pi(4k^2 - 1)^2} \sin 2k\theta.]$$

- B2.** State Stokes's theorem.

Evaluate $\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, dS$ for $\mathbf{A} = yz\mathbf{i} + xz\mathbf{k}$ and the surface S of an cylindrical container, open at the top, which has two parts, $x^2 + y^2 = 4$, $0 \leq z \leq 3$ and $z = 0$, $x^2 + y^2 \leq 4$,

- by direct integration (you may find it useful to recall that x , y expressed in cylindrical polar coordinates are $x = r \cos \theta$, $y = r \sin \theta$), taking the normal outward from the container, and
- by using Stokes's theorem.
- Check agreement with one aspect of Stokes's theorem by directly calculating $\int_S (\nabla \times \mathbf{A}) \cdot \mathbf{n} \, dS$ over the disk $z = 3$, $x^2 + y^2 \leq 4$ and showing that with an appropriate direction for $\mathbf{n}$ it agrees with the previous answer.

B3. Using Frobenius' method, find two independent solutions of the differential equation

$$4xy'' + 2y' + y = 0,$$

of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+\sigma}.$$

Give the first four terms of each series.

Find the radius of convergence of these series.

B4. Starting from Maxwell's equations,

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho/\epsilon_0, & \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, & \nabla \times \mathbf{B} &= \mu_0 \mathbf{j} + \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}, \end{aligned}$$

(where $c^2 = 1/(\epsilon_0 \mu_0)$, ϵ_0 and μ_0 are constants), show that

- (a) $\partial \rho / \partial t + \nabla \cdot \mathbf{j} = 0$ (charge conservation equation);
- (b) $\int_S \mathbf{E} \cdot \mathbf{n} \, dS = \int_V \rho / \epsilon_0 \, dV$ (Gauss's theorem);
- (c) $\int_S \mathbf{B} \cdot \mathbf{n} \, dS = 0$.

In (b) and (c), V is a closed volume and S is its surface.

An infinitely long straight wire of constant radius a lies along the z -axis of a cylindrical polar coordinate system and carries a uniformly distributed, steady current of magnitude I . By evaluating the integral $\int_C \mathbf{B} \cdot d\mathbf{r}$ around a suitable closed path C , show that the θ component of the associated magnetic field $\mathbf{B}$ is

$$B_\theta = \frac{\mu_0 I}{2\pi r},$$

where r is the distance from the wire. (You may wish to employ Stokes' Theorem and the time-independent Ampere Law, $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$, where $\mathbf{j}$ is the current density).